Examples of non-uniruled surfaces with pre-Tango structures involving non-closed global differential 1-forms

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Abstract

The pre-Tango structure is an ample invertible sheaf of locally exact differentials on a variety in positive characteristic, which often brings various sorts of pathological phenomena. We, however, know few examples of pre-Tango structures on non-uniruled varieties. In the present article, we explicitly construct non-uniruled surfaces with pre-Tango structures involving non-closed global differential 1-forms.

1 Introduction

Let $X$ be a projective algebraic variety over an algebraically closed field $k$ of characteristic $p > 0$ and let $F_X : X \to X$ be the relative Frobenius morphism over $k$. We then have a short exact sequence

$$0 \to \mathcal{O}_X \to F_X^* \mathcal{O}_X \to F_X^* \mathcal{B}_1^X \to 0,$$

where $\mathcal{B}_1^X$ is the first sheaf of coboundaries of the de Rham complex of $X$. Suppose that there exists an ample invertible subsheaf $\mathcal{L}$ of $F_X^* \mathcal{B}_1^X$ provided that $F_X^* \mathcal{B}_1^X$ is regarded as an $\mathcal{O}_X$-module. We call $\mathcal{L}$ a pre-Tango structure (see Takeda [10], see also Mukai [4]). Let us consider the exact sequence

$$0 \to \mathcal{L}^{-1} \to F_X^* \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \to F_X^* \mathcal{B}_1^X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \to 0.$$

By taking cohomology, we have

$$0 \to H^0(X, \mathcal{L}^{-1}) \to H^0(X, F_X^* \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) \to H^0(X, F_X^* \mathcal{B}_1^X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1})$$

$$\to H^1(X, \mathcal{L}^{-1}) \to \cdots.$$

Since $H^0(X, F_X^* \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) = 0$ and $H^0(X, F_X^* \mathcal{B}_1^X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) \neq 0$, we know that $H^1(X, \mathcal{L}^{-1}) \neq 0$. Hence, if $X$ is a smooth variety of dimension greater than one, then the pair $(X, \mathcal{L})$ is a counter-example to the Kodaira vanishing theorem in positive characteristic. It is, however, hard to find such a pair in dimension greater than one. Meanwhile, regarding in dimension one, we know that almost all smooth projective curves have pre-Tango structures (see Takeda and Yokogawa [11]). In fact, Raynaud’s famous counter-example ([7]) is a uniruled surface constructed by using a certain pre-Tango structure on a smooth projective curve.

The uniruled surfaces which are constructed similarly to Raynaud’s method are the only known examples of smooth surfaces which have pre-Tango structures, as far as the author knows. Hence the following problem seems interesting:

Suppose that a smooth projective surface $X$ has a pre-Tango structure. Then is $X$ a uniruled surface?

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Regrettably, the author does not know what the answer is. Meanwhile, it is known that, if a smooth non-uniruled projective variety \( X \) has an ample invertible sheaf \( L \) such that \( L^{p-1} \otimes \mathcal{O}_X \omega_X^{-1} \) is ample, then we have \( H^1(X, \mathcal{L}^{-1}) = 0 \) (Corollary II.6.3 in Kollár [3]). On the other hand, in case of normal projective varieties, the answer is negative. Indeed, Mumford gave an example of a pre-Tango structure on a normal projective surface, which is not uniruled ([6]). It seems, however, hard to know whether its desingularization has a pre-Tango structure or not.

For any smooth proper variety over \( k \) which lifts over the ring of Witt-vectors of length 2, the Kodaira vanishing theorem holds on it. Furthermore, if it is of dimension \( \leq p \), then its spectral sequence of Hodge to de Rham degenerates at \( E_1 \) (Deligne and Illusie [1]). So, it has no non-closed global differential 1-forms. In other words, the existence of non-closed global differential 1-forms is another typical pathological phenomenon in positive characteristic. Meanwhile, we know that, if a normal projective variety has non-closed global differential 1-forms, then so does its desingularization. Therefore, it seems appropriate to investigate normal projective surfaces with pre-Tango structures involving non-closed global differential 1-forms for the first step. In fact, we often see the normal uniruled surfaces, which are constructed similarly to Raynaud’s method by using pre-Tango structures on curves, having non-closed global differential 1-forms (cf. [11]).

On the other hand, it is well-known that we can easily construct surfaces with non-closed global differential 1-forms by using Mumford’s method, that is, by taking the composite of many Artin-Schreier coverings of base surfaces ([5]). We, however, hardly know their properties because of its elusive construction. Under the circumstances, the purpose of the present article is to give explicit and concrete examples of non-uniruled normal surfaces with pre-Tango structures involving non-closed global differential 1-forms in characteristic 2, 3. Precisely, we first consider a certain quotient of a superspecial abelian surface (the product of two supersingular elliptic curves) and take the composite finite covering of two suitable Artin-Schreier coverings of the quotient. On that finite covering, then we find out a pre-Tango structure with required attribute.

2 Case of characteristic \( p = 2 \)

2.1 A rational vector field on an abelian surface and the quotient

Let \( E_1 \) be the elliptic curve defined by

\[
y^2 + y = x^3,
\]

which is the unique supersingular elliptic curve in characteristic 2. We then have

\[
z + z^2 = w^3
\]

near the point at infinity, where \( z = y^{-1} \) and \( w = xy^{-1} \). Moreover, we have

\[
dy = x^2 dx \quad \text{and} \quad dz = w^2 dw.
\]

Note that

\[
x + w = x + \frac{x}{y} = \frac{x(y^2 + y)}{y^2} = \frac{x \cdot x^3}{y^2} = x^2 \frac{x^2}{y^2} = x^2 w^2.
\]

We then know

\[
dx = dw \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial w}.
\]

Take a copy \( \tilde{E} \) of \( E_1 \) and take the local parameters \( \xi_0 \) and \( \xi_\infty \) corresponding to \( x \) and \( w \), respectively. We then have the same equations

\[
\xi_0 + \xi_\infty = \xi^2_0 \xi^2_\infty, \quad d\xi_0 = d\xi_\infty, \quad \frac{\partial}{\partial \xi_0} = \frac{\partial}{\partial \xi_\infty}.
\]
as above. Let $A$ be the product $E_1 \times \tilde{E}$ and consider the $p$-closed rational vector field

$$D = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial z} \quad (i = 0, \infty)$$

on $A$. We know that

$$D = \frac{1}{z^2} \left( z^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \right) \quad (i = 0, \infty)$$

and that the divisor of $D$ is

$$(D) = -6S,$$

where $S$ is the fibre of the point at infinity of $E_1$, in other words, the curve defined by $w = 0$ on $A$. Besides $S$ is an integral curve of $D$.

Take the quotient $X$ of $A$ by $D$, i.e., the underlying topological space is the same as $A$ and the structure sheaf is the sheaf of the germs killed by $D$ (see Rudakov and Shafarevich [8]). Since $D$ has only divisorial singularities, we have that $X$ is a nonsingular surface of Kodaira dimension 1 (see Katsura and Takeda [2]). Let $\Gamma$ and $\Sigma$ denote the images by the quotient morphism of $S$ (the same as above) and $T = \{x = 0\}$, respectively. Since $S$ is an integral curve of $D$ and $T$ is not, we have that $[k(S):k(\Gamma)] = 2$ and $[k(T):k(\Sigma)] = 1$. Consider the relative Frobenius morphisms $F_2: \tilde{E} \to E$ and $F_1: E_1 \to E_1^{(p)}$ over $k$. We then have two fibrations: one is an elliptic fibration $\psi: X \to E_1^{(p)}$ induced from the first projection $A \to E_1$; and the other is a fibration $\phi: X \to E$ induced from the second projection $A \to \tilde{E}$, each fibre of which is an elliptic curve with one cusp. By regarding the fibration $\psi$, we know that $\Gamma$ is a fibre of multiplicity 2 and that $\Sigma$ is a fibre of multiplicity 1, and by regarding the fibration $\phi$, we know that $\Gamma$ is a section and that $\Sigma$ is a 2-section.

Let us consider local defining equations of $X$. Set $\xi_i = \xi_i^2$ for $i = 0, \infty$ and take the affine open subsets $U_0 = E - \{\eta_\infty = 0\}$, $U_\infty = E - \{\eta_0 = 0\}$. Take, furthermore, the affine open subsets

$$V_i = \phi^{-1}(U_i) - \Gamma, \quad W_i = \phi^{-1}(U_i) - \Sigma \quad (i = 0, \infty)$$

of $X$. Since $y^2 + y = x^3$, we know $\frac{\partial y}{\partial x} = x^2$. Hence we have

$$D(xy + \xi_i) = \left( \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial z} \right)(xy + \xi_i) = y + x \cdot x^2 + y^2 = 0 \quad (i = 0, \infty).$$

Set $u = x^2$, $v = y^2$ and $t_i = xy + \xi_i$ for $i = 0, \infty$. We then know that $u, v, t_i \in k(X)$, $v^2 + v = u^3$, $t_0 + t_\infty = \eta_0 \eta_\infty$ and

$$t_i^2 = uv + \eta_i,$$

which are local defining equations of $V_i$ for $i = 0, \infty$. Next set $r = u^2$, $s = z^2$. We then know that $r, s \in k(X)$, $s + s^2 = r^3$, $s = v^{-1}$, $u + r = u^2 r^2$ and $t_i^2 s^2 = r + \eta_i s^2$ by simple calculation. Therefore, by setting $q_i = t_i s$, we have local defining equations

$$q_i^2 = r + \eta_i s^2$$

of $W_i$ for $i = 0, \infty$. By exterior differentiation on $X$, we obtain the relations:

$$du = dr, \quad dr = s^2 d\eta_i \quad (i = 0, \infty).$$

Let us consider the exact differential 1-form $\omega = du$. We then know that $\omega$ is regular on $V_i$ for $i = 0, \infty$. Moreover, since $\omega = s^2 d\eta_i$ for $i = 0, \infty$, we know that $\omega$ is regular on $W_i$ for $i = 0, \infty$. Therefore, we have that

$$\omega \in H^0(X, \mathcal{O}_X^{\frac{1}{2}}).$$

Since $s(1 + s) = r^3$ and $\Gamma$ is a fibre of multiplicity 2, we know that $(s^2) = 12 \Gamma$. Hence the divisor of $\omega$ is

$$(\omega) = 12 \Gamma.$$
and that implies an inclusion
\[ \mathcal{O}_X(12\Gamma)\omega \hookrightarrow \mathcal{R}_X. \]

By taking its adjoint, we obtain an injection
\[ \mathcal{O}_X(6\Gamma) \hookrightarrow F_X, \mathcal{R}_X. \]

It is, however, not a pre-Tango structure because \( \Gamma \) is not ample.

### 2.2 A pre-Tango structure on a finite covering of the quotient

Let \( P_0 \) be the point defined by \( \eta_0 = 0 \) on \( E \), and set \( H = \varphi^{-1}(P_0) \). Consider the finite extension field \( k(X)(\theta, \zeta) \) subjected to
\[ \theta^2 + \eta_0^2\theta = u \quad \text{and} \quad \zeta^2 + \eta_0^2\zeta = \eta_0 \]
and take the normalization \( \sigma : Y \to X \) in \( k(X)(\theta, \zeta) \). We then know that \( Y \) is not a uniruled surface. Furthermore, we have
\[ \eta_0^2d\theta = du, \quad \eta_0^2d\zeta = d\eta_0 \]
on \( Y \). Since \( \omega = du = s^2d\eta_0 \), we obtain
\[ \eta_0^2d\theta = s^2\eta_0^2d\zeta. \]

By regarding on \( Y \), we have
\[ (\omega) = \sigma^*(12\Gamma + 2H). \]

That induces an inclusion
\[ \mathcal{O}_Y(\sigma^*(12\Gamma + 2H))\omega \hookrightarrow \mathcal{R}_Y^1. \]

By taking its adjoint, we obtain an injection
\[ \mathcal{O}_Y(\sigma^*(6\Gamma + H)) \hookrightarrow F_Y, \mathcal{R}_Y^1. \]

Moreover, it is a pre-Tango structure because \( 6\Gamma + H \) is ample on \( X \) and so is \( \sigma^*(6\Gamma + H) \) on \( Y \).

Consider the differential 1-forms \( d\theta \) (which is exact) and \( t_0d\theta \) (which is not closed) on \( Y \). We have
\[ d\theta = s^2d\zeta \quad \text{and} \quad t_0d\theta = q_{0sd}\zeta. \]

Since \( d\theta \) and \( t_0d\theta \) are regular on \( \sigma^{-1}(V_0) \), and \( s^2d\zeta \) and \( q_{0sd}\zeta \) are so on \( \sigma^{-1}(W_0) \), we have
\[ d\theta, t_0d\theta \in H^0(\sigma^{-1}(\varphi^{-1}(U_0)), \Omega_Y^1) = H^0(U_0, \varphi_\ast\sigma_\ast\Omega_Y^1). \]

On the other hand, since \( \omega = du \) is regular on \( X \), we have that \( du \) is regular on \( \sigma^{-1}(\varphi^{-1}(U_\infty)) \). Hence we obtain that
\[ du \in H^0(\sigma^{-1}(\varphi^{-1}(U_\infty)), \mathcal{R}_Y^1) = H^0(U_\infty, \varphi_\ast\sigma_\ast\Omega_Y^1). \]

Next consider \( t_\infty\omega \). We then know that \( t_\infty\omega = t_\infty du \) is regular on \( V_\infty \). Besides, since \( t_\infty du = q_{\infty sd}\eta_\infty \), we know that \( t_\infty\omega \) is regular on \( W_\infty \). By regarding on \( Y \), we have
\[ t_\infty du \in H^0(\sigma^{-1}(\varphi^{-1}(U_\infty)), \Omega_Y^1) = H^0(U_\infty, \varphi_\ast\sigma_\ast\Omega_Y^1). \]

Now let us consider \( \mathcal{O}_E \)-submodules \( \mathcal{R} \) and \( \mathcal{J} \) of \( \varphi_\ast\sigma_\ast\Omega_Y^1 \) such that
\[
\begin{align*}
\mathcal{R}|_{U_0} &= \mathcal{O}_E|_{U_0}d\theta, & \mathcal{R}|_{U_\infty} &= \mathcal{O}_E|_{U_\infty}du, \\
\mathcal{J}|_{U_0} &= \mathcal{O}_E|_{U_0}d\theta + \mathcal{O}_E|_{U_0}t_0d\theta, & \mathcal{J}|_{U_\infty} &= \mathcal{O}_E|_{U_\infty}du + \mathcal{O}_E|_{U_\infty}t_\infty du.
\end{align*}
\]
Note that the sections of \( \mathcal{S} \) which are not contained in \( \mathcal{R} \), are non-closed differential 1-forms. Meanwhile, since \( \eta_2^2 d\theta = du \), we know that \( \mathcal{R} \cong \mathcal{O}_E(2P_0) \). Moreover, since \( t_0 + t_\infty = \eta_0 \eta_\infty \), we have

\[
\eta_2^2 t_0 d\theta = t_\infty du + \eta_0 \eta_\infty du
\]

and so,

\[
(d\theta, t_0 d\theta) = (du, t_\infty du)
\]

\[
\left(\frac{1}{\eta_0}, \frac{\eta_\infty}{\eta_0}, 0, \frac{1}{\eta_0^2}\right).
\]

Therefore, we obtain a short exact sequence

\[
0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{O}_E(2P_0) \to 0.
\]

Since \( H^1(E, \mathcal{R}) \cong H^1(E, \mathcal{O}_E(2P_0)) = 0 \) and \( H^0(E, \mathcal{O}_E(2P_0)) \neq 0 \), we know \( H^0(E, \mathcal{R}) \cong H^0(E, \mathcal{S}) \). Hence we conclude that \( Y \) has non-closed global differential 1-forms.

### 3 Case of characteristic \( p = 3 \)

#### 3.1 A rational vector field on an abelian surface and the quotient

Let \( E_1 \) be the elliptic curve defined by

\[
y^2 = x^3 - x,
\]

which is the unique supersingular elliptic curve in characteristic 3. We then have

\[
z = w^3 - wz^2
\]

near the point at infinity, where \( z = y^{-1} \) and \( w = xy^{-1} \). Moreover, we know \( 2ydy = -dx \) and so

\[
dy = \frac{dx}{y},
\]

which is an exact global differential 1-form. Set \( \Delta = y \frac{\partial}{\partial x} \). We then have that \( \Delta \) is a regular vector field on \( E_1 \) such that \( \Delta^3 = 0 \). Note that

\[
z = w^3 - wz^2
\]

\[
z(1 + wz) = w^3
\]

\[
z \left( \frac{1}{w^3} + \frac{1}{w^3} wz \right) = 1
\]

\[
\frac{1}{w^3} + \frac{1}{w^3} wz = y
\]

\[
\frac{1}{w^3} + \frac{1}{w^3} w \frac{w^3}{1 + wz} = y
\]

\[
\frac{1}{w^3} + \frac{w}{1 + wz} = y.
\]

We then obtain that

\[
dy = d \frac{w}{1 + wz}.
\]
Meanwhile, we know that \( y \) (resp. \( w/(1 + wz) \)) is a local parameter near the point over \( x = 0 \) (resp. \( x = \infty \)).

Take a copy \( \tilde{E} \) of \( E_1 \) and take the local parameters \( \xi_0 \) and \( \xi_{\infty} \) corresponding to \( y \) and \( w/(1 + wz) \), respectively. We then have

\[
d\xi_0 = d\xi_{\infty} \quad \text{and} \quad \frac{\partial}{\partial \xi_0} = \frac{\partial}{\partial \xi_{\infty}}.
\]

Let \( A \) be the product \( E_1 \times \tilde{E} \) and consider the \( p \)-closed rational vector field

\[
D = \Delta - x^3 \frac{\partial}{\partial \xi_i} \quad (i = 0, \infty)
\]
on \( A \). We know that

\[
D = \frac{1}{\xi^2} \left( \xi^2 \Delta - (1 + wz) \frac{\partial}{\partial \xi_i} \right) \quad (i = 0, \infty)
\]

and that the divisor of \( D \) is

\[
(D) = -6S,
\]

where \( S \) is the fibre of the point at infinity of \( E_1 \), in other words, the curve defined by \( w = 0 \) on \( A \). Besides \( S \) is an integral curve of \( D \).

Take the quotient \( X \) of \( A \) by \( D \), i.e., the underlying topological space is the same as \( A \) and the structure sheaf is the sheaf of the germs killed by \( D \) (see [8]). Since \( D \) has only divisorial singularities, we have that \( X \) is a nonsingular surface of Kodaira dimension 1 (see [2]). Let \( \Gamma \) and \( \Sigma \) denote the images by the quotient morphism of \( S \) (the same as above) and \( T = \{ x = 0 \} \), respectively. Since \( S \) is an integral curve of \( D \) and \( T \) is not, we have that \( [k(S) : k(\Gamma)] = 3 \) and \( [k(T) : k(\Sigma)] = 1 \). Consider the relative Frobenius morphisms \( F_\xi : \tilde{E} \to E \) and \( F_1 : E_1 \to E_1^{(p)} \) over \( k \). We then have two fibrations: one is an elliptic fibration \( \psi : X \to E_1^{(p)} \) induced from the first projection \( A \to E_1 \); and the other is a fibration \( \varphi : X \to E \) induced from the second projection \( A \to \tilde{E} \), each fibre of which is an elliptic curve with one cusp. By regarding the fibration \( \psi \), we know that \( \Gamma \) is a fibre of multiplicity 3 and that \( \Sigma \) is a fibre of multiplicity 1, and by regarding the fibration \( \varphi \), we know that \( \Gamma \) is a section and that \( \Sigma \) is a 3-section.

Let us consider local defining equations of \( X \). Set \( \eta_i = \xi^3 \) for \( i = 0, \infty \) and take the affine open subsets \( U_0 = E - \{ \eta_\infty = 0 \}, U_\infty = E - \{ \eta_0 = 0 \} \). Take, furthermore, the affine open subsets

\[
V_i = \varphi^{-1}(U_i) - \Gamma, \quad W_i = \varphi^{-1}(U_i) - \Sigma \quad (i = 0, \infty)
\]
of \( X \). Since \( \Delta(y) = 1 \) and \( \Delta(x) = y \), we obtain

\[
D(xy + \xi_i) = \left( \Delta - x^3 \frac{\partial}{\partial \xi_i} \right)(xy + \xi_i) = y^2 + x - x^3 = 0 \quad (i = 0, \infty).
\]

Set \( u = x^3, v = y^3 \) and \( t_i = xy + \xi_i \) for \( i = 0, \infty \). We then know that \( u, v, t_i \in k(X), v^2 = u^3 - u \) and

\[
t_i^3 = uv + \eta_i,
\]

which are local defining equations of \( V_i \) for \( i = 0, \infty \). By exterior differentiation on \( X \), we obtain the relations:

\[
vdv = du, \quad u^3dv = -d\eta_i \quad (i = 0, \infty).
\]

Next set \( r = w^3, s = \xi^3 \). We then know that \( r, s \in k(X), s = r^3 - rs^2, s = v^{-1}, r = uv^{-1} \) and \( t_i^3s^3 = rs + \eta_is^3 \).

Therefore, by setting \( q_i = t_is \), we have local defining equations

\[
q_i^3 = rs + \eta_is^3
\]
of \( W_i \) for \( i = 0, \infty \).
Let us consider the exact differential 1-form $\omega = dv$. We then know that $\omega$ is regular on $V_i$ for $i = 0, \infty$. Moreover, by simple computation, we have $\omega = -\frac{s^2}{1+rs} d\eta_i$ for $i = 0, \infty$. Hence we know that $\omega$ is regular on $W_i$ for $i = 0, \infty$. Therefore, we have that $\omega \in H^0(X, B_X)$.

Since $s(1+rs) = r^3$ and $\Gamma$ is a fibre of multiplicity 3, we know that $(s^2) = 18\Gamma$. Hence the divisor of $\omega$ is $(\omega) = 18\Gamma$ and that implies an inclusion $\mathcal{O}_X(18\Gamma) \hookrightarrow B_X^{1}$.

By taking its adjoint, we obtain an injection

$$\mathcal{O}_X(6\Gamma) \hookrightarrow F_X^{*}B_X^{1}.$$  

It is, however, not a pre-Tango structure because $\Gamma$ is not ample.

### 3.2 A pre-Tango structure on a finite covering of the quotient

Let $P_0$ be the point defined by $\eta_0 = 0$ on $E$, and set $H = \varphi^{-1}(P_0)$. Consider the finite extension field $k(X)(\theta, \zeta)$ subjected to

$$\theta^3 - \eta_0^3\theta = v \quad \text{and} \quad \zeta^3 - \eta_0^3\zeta = \eta_0$$

and take the normalization $\sigma : Y \to X$ in $k(X)(\theta, \zeta)$. We then know that $Y$ is not a uniruled surface. Furthermore, we have

$$-\eta_0^3d\theta = dv, \quad -\eta_0^3d\zeta = d\eta_0$$

on $Y$. Since $\omega = dv = -\frac{s^2}{1+rs} d\eta_0$, we obtain

$$\eta_0^3d\theta = -\frac{s^2}{1+rs} d\zeta.$$  

By regarding on $Y$, we have

$$(\omega) = \sigma^*(18\Gamma + 3H).$$

That induces an inclusion

$$\mathcal{O}_Y(\sigma^*(18\Gamma + 3H)) \hookrightarrow B_Y^{1}.$$  

By taking its adjoint, we obtain an injection

$$\mathcal{O}_Y(\sigma^*(6\Gamma + H)) \hookrightarrow F_Y^{*}B_Y^{1}.$$  

Moreover, it is a pre-Tango structure because $6\Gamma + H$ is ample on $X$ and so is $\sigma^*(6\Gamma + H)$ on $Y$.

Consider the differential 1-forms $d\theta$ (which is exact) and $t_0d\theta$ (which is not closed) on $Y$. We have

$$d\theta = -\frac{s^2}{1+rs} d\zeta \quad \text{and} \quad t_0d\theta = -\frac{q_0s}{1+rs} d\zeta.$$  

Since $d\theta, t_0d\theta$ are regular on $\sigma^{-1}(V_0)$ and since $\frac{s^2}{1+rs} d\zeta, \frac{q_0s}{1+rs} d\zeta$ are so on $\sigma^{-1}(W_0)$, we have

$$d\theta, t_0d\theta \in H^0(\sigma^{-1}(U_0), \Omega_Y^{1}) = H^0(U_0, \varphi_*\sigma_*\Omega_Y^{1}).$$
On the other hand, since \( \omega = dv \) is regular on \( X \), we have that \( dv \) is regular on \( \sigma^{-1}(\phi^{-1}(U_\infty)) \). Hence we obtain that
\[
dv \in H^0(\sigma^{-1}(\phi^{-1}(U_\infty)), \mathcal{B}_Y^1) = H^0(U_\infty, \phi_\ast \sigma_\ast \mathcal{B}_Y^1).
\]
Next consider \( t_\infty \omega \). We then know that \( t_\infty \omega = t_\infty dv \) is regular on \( V_\infty \). Besides, since \( t_\infty dv = -\frac{q_{i=0}^{\infty}}{1 + r s} d\eta_\infty \), we know that \( t_\infty \omega \) is regular on \( W_\infty \). By regarding on \( Y \), we have
\[
t_\infty dv \in H^0(\sigma^{-1}(\phi^{-1}(U_\infty)), \Omega_Y^1) = H^0(U_\infty, \phi_\ast \sigma_\ast \Omega_Y^1).
\]
Now let us consider \( E \)-submodules \( \mathcal{R} \) and \( \mathcal{I} \) of \( \phi_\ast \sigma_\ast \Omega_Y^1 \) such that
\[
\mathcal{R}|_{U_0} = \mathcal{E}|_{U_0} d\theta, \quad \mathcal{R}|_{U_\infty} = \mathcal{E}|_{U_\infty} dv,
\]
\[
\mathcal{I}|_{U_0} = \mathcal{E}|_{U_0} d\theta + \mathcal{E}|_{U_0} t_0 d\theta, \quad \mathcal{I}|_{U_\infty} = \mathcal{E}|_{U_\infty} dv + \mathcal{E}|_{U_\infty} t_\infty dv.
\]
Note that the sections of \( \mathcal{I} \) which are not contained in \( \mathcal{R} \), are non-closed differential 1-forms. Meanwhile, since \( -\eta_0^3 d\theta = dv \), we know that \( \mathcal{R} \cong \mathcal{E}(3P_0) \). Recall that \( \xi, \xi_\infty \) are corresponding to \( y, w/(1 + wz) \), respectively. Therefore, the difference \( \xi_0 - \xi_\infty \) is corresponding to \( 1/w^3 \). Denote it by \( b_{0\infty} \). We then have that \( b_{0\infty} \) is a section in \( \mathcal{E}(U_0 \cap U_\infty) \). Moreover, since \( t_i = xy + \xi_i \) for \( i = 0, \infty \), we know that \( t_0 - t_\infty = b_{0\infty} \). Hence we have
\[
-\eta_0^3 t_0 d\theta = t_\infty dv + b_{0\infty} dv
\]
and so,
\[
(-d\theta, -t_0 d\theta) = (dv, t_\infty dv)
\]
\[
\begin{pmatrix}
\frac{1}{\eta_0^3} & \frac{b_{0\infty}}{\eta_0^3} \\
0 & 1/\eta_0^3
\end{pmatrix}.
\]
Therefore, we obtain a short exact sequence
\[
0 \to \mathcal{R} \to \mathcal{I} \to \mathcal{E}(3P_0) \to 0.
\]
Since \( H^1(E, \mathcal{R}) \cong H^1(E, \mathcal{E}(3P_0)) = 0 \) and \( H^0(E, \mathcal{E}(3P_0)) \neq 0 \), we know \( H^0(E, \mathcal{R}) \subsetneq H^0(E, \mathcal{I}) \). Hence we conclude that \( Y \) has non-closed global differential 1-forms.

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References


