Examples of non-uniruled surfaces with pre-Tango structures involving non-closed global differential 1-forms

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Abstract

The pre-Tango structure is an ample invertible sheaf of locally exact differentials on a variety in positive characteristic, which often brings various sorts of pathological phenomena. We, however, know few examples of pre-Tango structures on non-uniruled varieties. In the present article, we explicitly construct non-uniruled surfaces with pre-Tango structures involving non-closed global differential 1-forms.

1 Introduction

Let X be a projective algebraic variety over an algebraically closed field k of characteristic $p > 0$ and let $F_X : \widetilde{X} \to X$ be the relative Frobenius morphism over k. We then have a short exact sequence

$$0 \to \mathscr{O}_X \to F_{X*}\mathscr{O}_{\widetilde{X}} \to F_{X*}\mathscr{B}^1_{\widetilde{X}} \to 0,$$

where $\mathscr{B}^1_{\widetilde{X}}$ is the first sheaf of coboundaries of the de Rham complex of $\widetilde{X}$. Suppose that there exists an ample invertible subsheaf $\mathscr{L}$ of $F_{X*}\mathscr{B}^1_{\widetilde{X}}$ provided that $F_{X*}\mathscr{B}^1_{\widetilde{X}}$ is regarded as an $\mathscr{O}_X$-module. We call $\mathscr{L}$ a *pre-Tango structure* (see Takeda [10], see also Mukai [4]). Let us consider the exact sequence

$$0 \to \mathscr{L}^{-1} \to F_{X*}\mathscr{O}_{\widetilde{X}} \otimes_{\mathscr{O}_X} \mathscr{L}^{-1} \to F_{X*}\mathscr{B}^1_{\widetilde{X}} \otimes_{\mathscr{O}_X} \mathscr{L}^{-1} \to 0.$$

By taking cohomology, we have

$$\begin{aligned} 0 &\to H^0(X,\mathscr{L}^{-1}) \to H^0(X, F_{X*}\mathscr{O}_{\widetilde{X}} \otimes_{\mathscr{O}_X} \mathscr{L}^{-1}) \to H^0(X, F_{X*}\mathscr{B}^1_{\widetilde{X}} \otimes_{\mathscr{O}_X} \mathscr{L}^{-1}) \\ &\to H^1(X,\mathscr{L}^{-1}) \to \cdots \end{aligned}$$

Since $H^0(X, F_{X*}\mathscr{O}_{\widetilde{X}} \otimes_{\mathscr{O}_X} \mathscr{L}^{-1}) = 0$ and $H^0(X, F_{X*}\mathscr{B}^1_{\widetilde{X}} \otimes_{\mathscr{O}_X} \mathscr{L}^{-1}) \neq 0$, we know that $H^1(X,\mathscr{L}^{-1}) \neq 0$. Hence, if X is a smooth variety of dimension greater than one, then the pair $(X,\mathscr{L})$ is a counter-example to the Kodaira vanishing theorem in positive characteristic. It is, however, hard to find such a pair in dimension greater than one. Meanwhile, regarding in dimension one, we know that almost all smooth projective curves have pre-Tango structures (see Takeda and Yokogawa [11]). In fact, Raynaud's famous counter-example ([7]) is a uniruled surface constructed by using a certain pre-Tango structure on a smooth projective curve.

The uniruled surfaces which are constructed similarly to Raynaud's method are the only known examples of smooth surfaces which have pre-Tango structures, as far as the author knows. Hence the following problem seems interesting:

> Suppose that a smooth projective surface X has a pre-Tango structure. Then is X a uniruled surface?

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Regrettably, the author does not know what the answer is. Meanwhile, it is known that, if a smooth non-uniruled projective variety X has an ample invertible sheaf $\mathscr{L}$ such that $\mathscr{L}^{p-1} \otimes_{\mathscr{O}_X} \omega_X^{-1}$ is ample, then we have $H^1(X, \mathscr{L}^{-1}) = 0$ (Corollary II.6.3 in Kollár [3]). On the other hand, in case of normal projective varieties, the answer is negative. Indeed, Mumford gave an example of a pre-Tango structure on a normal projective surface, which is not uniruled ([6]). It seems, however, hard to know whether its desingularization has a pre-Tango structure or not.

For any smooth proper variety over k which lifts over the ring of Witt-vectors of length 2, the Kodaira vanishing theorem holds on it. Furthermore, if it is of dimension $\leq p$, then its spectral sequence of Hodge to de Rham degenerates at E_1 (Deligne and Illusie [1]). So, it has no non-closed global differential 1-forms. In other words, the existence of non-closed global differential 1-forms is another typical pathological phenomenon in positive characteristic. Meanwhile, we know that, if a normal projective variety has non-closed global differential 1-forms, then so does its desingularization. Therefore, it seems appropriate to investigate normal projective surfaces with pre-Tango structures involving non-closed global differential 1-forms for the first step. In fact, we often see the normal uniruled surfaces, which are constructed similarly to Raynaud's method by using pre-Tango structures on curves, having non-closed global differential 1-forms (cf. [11]).

On the other hand, it is well-known that we can easily construct surfaces with non-closed global differential 1-forms by using Mumford's method, that is, by taking the composite of many Artin-Schreier coverings of base surfaces ([5]). We, however, hardly know their properties because of its elusive construction. Under the circumstances, the purpose of the present article is to give explicit and concrete examples of non-uniruled normal surfaces with pre-Tango structures involving non-closed global differential 1-forms in characteristic 2, 3. Precisely, we first consider a certain quotient of a superspecial abelian surface (the product of two supersingular elliptic curves) and take the composite finite covering of *two* suitable Artin-Schreier coverings of the quotient. On that finite covering, then we find out a pre-Tango structure with required attribute.

2 Case of characteristic $p = 2$

2.1 A rational vector field on an abelian surface and the quotient

Let E_1 be the elliptic curve defined by

$$y^2 + y = x^3,$$

which is the unique supersingular elliptic curve in characteristic 2. We then have

$$z + z^2 = w^3$$

near the point at infinity, where $z = y^{-1}$ and $w = xy^{-1}$. Moreover, we have

$$dy = x^2 dx \quad \text{and} \quad dz = w^2 dw.$$

Note that

$$x + w = x + \frac{x}{y} = \frac{x(y^2+y)}{y^2} = \frac{x \cdot x^3}{y^2} = x^2 \frac{x^2}{y^2} = x^2 w^2.$$

We then know

$$dx = dw \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial w}.$$

Take a copy $\widetilde{E}$ of E_1 and take the local parameters ξ_0 and ξ_∞ corresponding to x and w, respectively. We then have the same equations

$$\xi_0 + \xi_\infty = \xi_0^2 \xi_\infty^2, \quad d\xi_0 = d\xi_\infty, \quad \frac{\partial}{\partial \xi_0} = \frac{\partial}{\partial \xi_\infty}$$

as above. Let A be the product $E_1 \times \widetilde{E}$ and consider the p-closed rational vector field

$$D = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial \xi_i} \quad (i = 0, \infty)$$

on A. We know that

$$D = \frac{1}{z^2}\Big(z^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial \xi_i}\Big) \quad (i = 0, \infty)$$

and that the divisor of D is

$$(D) = -6S,$$

where S is the fibre of the point at infinity of E_1, in other words, the curve defined by $w = 0$ on A. Besides S is an integral curve of D.

Take the quotient X of A by D, i.e., the underlying topological space is the same as A and the structure sheaf is the sheaf of the germs killed by D (see Rudakov and Shafarevich [8]). Since D has only divisorial singularities, we have that X is a nonsingular surface of Kodaira dimension 1 (see Katsura and Takeda [2]). Let Γ and Σ denote the images by the quotient morphism of S (the same as above) and $T = \{x = 0\}$, respectively. Since S is an integral curve of D and T is not, we have that $[k(S) : k(\Gamma)] = 2$ and $[k(T) : k(\Sigma)] = 1$. Consider the relative Frobenius morphisms $F_E : \widetilde{E} \to E$ and $F_1 : E_1 \to E_1^{(p)}$ over k. We then have two fibrations: one is an elliptic fibration $\psi : X \to E_1^{(p)}$ induced from the first projection $A \to E_1$; and the other is a fibration $\varphi : X \to E$ induced from the second projection $A \to \widetilde{E}$, each fibre of which is an elliptic curve with one cusp. By regarding the fibration ψ, we know that Γ is a fibre of multiplicity 2 and that Σ is a fibre of multiplicity 1, and by regarding the fibration φ, we know that Γ is a section and that Σ is a 2-section.

Let us consider local defining equations of X. Set $\eta_i = \xi_i^2$ for $i = 0, \infty$ and take the affine open subsets $U_0 = E - \{\eta_\infty = 0\}$, $U_\infty = E - \{\eta_0 = 0\}$. Take, furthermore, the affine open subsets

$$V_i = \varphi^{-1}(U_i) - \Gamma, \quad W_i = \varphi^{-1}(U_i) - \Sigma \quad (i = 0, \infty)$$

of X. Since $y^2 + y = x^3$, we know $\dfrac{\partial y}{\partial x} = x^2$. Hence we have

$$D(xy + \xi_i) = \Big(\frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial \xi_i}\Big)(xy + \xi_i) = y + x \cdot x^2 + y^2 = 0 \quad (i = 0, \infty).$$

Set $u = x^2$, $v = y^2$ and $t_i = xy + \xi_i$ for $i = 0, \infty$. We then know that $u, v, t_i \in k(X)$, $v^2 + v = u^3$, $t_0 + t_\infty = \eta_0 \eta_\infty$ and

$$t_i^2 = uv + \eta_i,$$

which are local defining equations of V_i for $i = 0, \infty$. Next set $r = w^2$, $s = z^2$. We then know that $r, s \in k(X)$, $s + s^2 = r^3$, $s = v^{-1}$, $u + r = u^2 r^2$ and $t_i^2 s^2 = r + \eta_i s^2$ by simple calculation. Therefore, by setting $q_i = t_i s$, we have local defining equations

$$q_i^2 = r + \eta_i s^2$$

of W_i for $i = 0, \infty$. By exterior differentiation on X, we obtain the relations:

$$du = dr, \quad dr = s^2 d\eta_i \quad (i = 0, \infty).$$

Let us consider the exact differential 1-form $\omega = du$. We then know that ω is regular on V_i for $i = 0, \infty$. Moreover, since $\omega = s^2 d\eta_i$ for $i = 0, \infty$, we know that ω is regular on W_i for $i = 0, \infty$. Therefore, we have that

$$\omega \in H^0(X, \mathscr{B}_X^1).$$

Since $s(1 + s) = r^3$ and Γ is a fibre of multiplicity 2, we know that $(s^2) = 12\Gamma$. Hence the divisor of ω is

$$(\omega) = 12\Gamma$$

and that implies an inclusion

$$\mathcal{O}_X(12\Gamma)\omega \hookrightarrow \mathcal{B}^1_X.$$

By taking its adjoint, we obtain an injection

$$\mathcal{O}_X(6\Gamma) \hookrightarrow F_{X*}\mathcal{B}^1_{\tilde{X}}.$$

It is, however, *not* a pre-Tango structure because Γ is not ample.

2.2 A pre-Tango structure on a finite covering of the quotient

Let P_0 be the point defined by $\eta_0 = 0$ on E, and set $H = \varphi^{-1}(P_0)$. Consider the finite extension field $k(X)(\theta,\zeta)$ subjected to

$$\theta^2 + \eta_0^2\theta = u \quad \text{and} \quad \zeta^2 + \eta_0^2\zeta = \eta_0$$

and take the normalization $\sigma : Y \to X$ in $k(X)(\theta,\zeta)$. We then know that Y *is not a uniruled surface*. Furthermore, we have

$$\eta_0^2 d\theta = du, \quad \eta_0^2 d\zeta = d\eta_0$$

on Y. Since $\omega = du = s^2 d\eta_0$, we obtain

$$\eta_0^2 d\theta = s^2\eta_0^2 d\zeta.$$

By regarding on Y, we have

$$(\omega) = \sigma^*(12\Gamma + 2H).$$

That induces an inclusion

$$\mathcal{O}_Y(\sigma^*(12\Gamma+2H))\omega \hookrightarrow \mathcal{B}^1_Y.$$

By taking its adjoint, we obtain an injection

$$\mathcal{O}_Y(\sigma^*(6\Gamma+H)) \hookrightarrow F_{Y*}\mathcal{B}^1_{\tilde{Y}}.$$

Moreover, it is a *pre-Tango structure* because $6\Gamma+H$ is ample on X and so is $\sigma^*(6\Gamma+H)$ on Y.

Consider the differential 1-forms $d\theta$ (which is exact) and $t_0 d\theta$ (which is not closed) on Y. We have

$$d\theta = s^2 d\zeta \quad \text{and} \quad t_0 d\theta = q_0 s d\zeta.$$

Since $d\theta$ and $t_0 d\theta$ are regular on $\sigma^{-1}(V_0)$, and $s^2 d\zeta$ and $q_0 s d\zeta$ are so on $\sigma^{-1}(W_0)$, we have

$$d\theta,\ t_0 d\theta \in H^0(\sigma^{-1}(\varphi^{-1}(U_0)), \Omega^1_Y) = H^0(U_0, \varphi_*\sigma_*\Omega^1_Y).$$

On the other hand, since $\omega = du$ is regular on X, we have that du is regular on $\sigma^{-1}(\varphi^{-1}(U_\infty))$. Hence we obtain that

$$du \in H^0(\sigma^{-1}(\varphi^{-1}(U_\infty)), \mathcal{B}^1_Y) = H^0(U_\infty, \varphi_*\sigma_*\mathcal{B}^1_Y).$$

Next consider $t_\infty\omega$. We then know that $t_\infty\omega = t_\infty du$ is regular on V_∞. Besides, since $t_\infty du = q_\infty s d\eta_\infty$, we know that $t_\infty\omega$ is regular on W_∞. By regarding on Y, we have

$$t_\infty du \in H^0(\sigma^{-1}(\varphi^{-1}(U_\infty)), \Omega^1_Y) = H^0(U_\infty, \varphi_*\sigma_*\Omega^1_Y).$$

Now let us consider $\mathcal{O}_E$-submodules $\mathcal{R}$ and $\mathcal{S}$ of $\varphi_*\sigma_*\Omega^1_Y$ such that

$$\mathcal{R}|_{U_0} = \mathcal{O}_E|_{U_0} d\theta, \qquad \mathcal{R}|_{U_\infty} = \mathcal{O}_E|_{U_\infty} du,$$

$$\mathcal{S}|_{U_0} = \mathcal{O}_E|_{U_0} d\theta + \mathcal{O}_E|_{U_0} t_0 d\theta, \qquad \mathcal{S}|_{U_\infty} = \mathcal{O}_E|_{U_\infty} du + \mathcal{O}_E|_{U_\infty} t_\infty du.$$

Note that the sections of $\mathscr{S}$ which are not contained in $\mathscr{R}$, are non-closed differential 1-forms. Meanwhile, since $\eta_0^2 d\theta = du$, we know that $\mathscr{R} \cong \mathscr{O}_E(2P_0)$. Moreover, since $t_0 + t_\infty = \eta_0\eta_\infty$, we have

$$\eta_0^2 t_0 d\theta = t_\infty du + \eta_0\eta_\infty du$$

and so,

$$(d\theta, t_0 d\theta) = (du, t_\infty du)\begin{pmatrix} \frac{1}{\eta_0^2} & \frac{\eta_\infty}{\eta_0} \\ 0 & \frac{1}{\eta_0^2} \end{pmatrix}.$$

Therefore, we obtain a short exact sequence

$$0 \to \mathscr{R} \to \mathscr{S} \to \mathscr{O}_E(2P_0) \to 0.$$

Since $H^1(E,\mathscr{R}) \cong H^1(E,\mathscr{O}_E(2P_0)) = 0$ and $H^0(E,\mathscr{O}_E(2P_0)) \neq 0$, we know $H^0(E,\mathscr{R}) \subsetneqq H^0(E,\mathscr{S})$. Hence we conclude that Y *has non-closed global differential* 1*-forms*.

3 Case of characteristic $p = 3$

3.1 A rational vector field on an abelian surface and the quotient

Let E_1 be the elliptic curve defined by

$$y^2 = x^3 - x,$$

which is the unique supersingular elliptic curve in characteristic 3. We then have

$$z = w^3 - wz^2$$

near the point at infinity, where $z = y^{-1}$ and $w = xy^{-1}$. Moreover, we know $2ydy = -dx$ and so

$$dy = \frac{dx}{y},$$

which is an exact global differential 1-form. Set $\Delta = y\frac{\partial}{\partial x}$. We then have that Δ is a regular vector field on E_1 such that $\Delta^3 = 0$. Note that

$$\begin{aligned} z &= w^3 - wz^2 \\ z(1+wz) &= w^3 \\ z\left(\frac{1}{w^3} + \frac{1}{w^3}wz\right) &= 1 \\ \frac{1}{w^3} + \frac{1}{w^3}wz &= y \\ \frac{1}{w^3} + \frac{1}{w^3}w\frac{w^3}{1+wz} &= y \\ \frac{1}{w^3} + \frac{w}{1+wz} &= y. \end{aligned}$$

We then obtain that

$$dy = d\frac{w}{1+wz}.$$

Meanwhile, we know that y (resp. $w/(1+wz)$) is a local parameter near the point over $x=0$ (resp. $x=\infty$).

Take a copy $\widetilde{E}$ of E_1 and take the local parameters ξ_0 and ξ_∞ corresponding to y and $w/(1+wz)$, respectively. We then have

$$d\xi_0 = d\xi_\infty \quad \text{and} \quad \frac{\partial}{\partial \xi_0} = \frac{\partial}{\partial \xi_\infty}.$$

Let A be the product $E_1 \times \widetilde{E}$ and consider the p-closed rational vector field

$$D = \Delta - x^3 \frac{\partial}{\partial \xi_i} \quad (i = 0, \infty)$$

on A. We know that

$$D = \frac{1}{z^2}\Big(z^2\Delta - (1+wz)\frac{\partial}{\partial \xi_i}\Big) \quad (i = 0, \infty)$$

and that the divisor of D is

$$(D) = -6S,$$

where S is the fibre of the point at infinity of E_1, in other words, the curve defined by $w=0$ on A. Besides S is an integral curve of D.

Take the quotient X of A by D, i.e., the underlying topological space is the same as A and the structure sheaf is the sheaf of the germs killed by D (see [8]). Since D has only divisorial singularities, we have that X is a nonsingular surface of Kodaira dimension 1 (see [2]). Let Γ and Σ denote the images by the quotient morphism of S (the same as above) and $T = \{x = 0\}$, respectively. Since S is an integral curve of D and T is not, we have that $[k(S) : k(\Gamma)] = 3$ and $[k(T) : k(\Sigma)] = 1$. Consider the relative Frobenius morphisms $F_E : \widetilde{E} \to E$ and $F_1 : E_1 \to E_1^{(p)}$ over k. We then have two fibrations: one is an elliptic fibration $\psi : X \to E_1^{(p)}$ induced from the first projection $A \to E_1$; and the other is a fibration $\varphi : X \to E$ induced from the second projection $A \to \widetilde{E}$, each fibre of which is an elliptic curve with one cusp. By regarding the fibration ψ, we know that Γ is a fibre of multiplicity 3 and that Σ is a fibre of multiplicity 1, and by regarding the fibration φ, we know that Γ is a section and that Σ is a 3-section.

Let us consider local defining equations of X. Set $\eta_i = \xi_i^3$ for $i = 0, \infty$ and take the affine open subsets $U_0 = E - \{\eta_\infty = 0\}$, $U_\infty = E - \{\eta_0 = 0\}$. Take, furthermore, the affine open subsets

$$V_i = \varphi^{-1}(U_i) - \Gamma, \quad W_i = \varphi^{-1}(U_i) - \Sigma \quad (i = 0, \infty)$$

of X. Since $\Delta(y) = 1$ and $\Delta(x) = y$, we obtain

$$D(xy + \xi_i) = \Big(\Delta - x^3\frac{\partial}{\partial \xi_i}\Big)(xy + \xi_i) = y^2 + x - x^3 = 0 \quad (i = 0, \infty).$$

Set $u = x^3$, $v = y^3$ and $t_i = xy + \xi_i$ for $i = 0, \infty$. We then know that $u, v, t_i \in k(X)$, $v^2 = u^3 - u$ and

$$t_i^3 = uv + \eta_i,$$

which are local defining equations of V_i for $i = 0, \infty$. By exterior differentiation on X, we obtain the relations:

$$vdv = du, \quad u^3dv = -d\eta_i \quad (i = 0, \infty).$$

Next set $r = w^3$, $s = z^3$. We then know that $r, s \in k(X)$, $s = r^3 - rs^2$, $s = v^{-1}$, $r = uv^{-1}$ and $t_i^3s^3 = rs + \eta_i s^3$. Therefore, by setting $q_i = t_i s$, we have local defining equations

$$q_i^3 = rs + \eta_i s^3$$

of W_i for $i = 0, \infty$.

Let us consider the exact differential 1-form $\omega = dv$. We then know that ω is regular on V_i for $i = 0, \infty$. Moreover, by simple computation, we have $\omega = -\dfrac{s^2}{1+rs} d\eta_i$ for $i = 0, \infty$. Hence we know that ω is regular on W_i for $i = 0, \infty$. Therefore, we have that

$$\omega \in H^0(X, \mathscr{B}^1_X).$$

Since $s(1+rs) = r^3$ and Γ is a fibre of multiplicity 3, we know that $(s^2) = 18\Gamma$. Hence the divisor of ω is

$$(\omega) = 18\Gamma$$

and that implies an inclusion

$$\mathscr{O}_X(18\Gamma)\omega \hookrightarrow \mathscr{B}^1_X.$$

By taking its adjoint, we obtain an injection

$$\mathscr{O}_X(6\Gamma) \hookrightarrow F_{X*}\mathscr{B}^1_{\tilde{X}}.$$

It is, however, *not* a pre-Tango structure because Γ is not ample.

3.2 A pre-Tango structure on a finite covering of the quotient

Let P_0 be the point defined by $\eta_0 = 0$ on E, and set $H = \varphi^{-1}(P_0)$. Consider the finite extension field $k(X)(\theta, \zeta)$ subjected to

$$\theta^3 - \eta_0^3\theta = v \quad \text{and} \quad \zeta^3 - \eta_0^3\zeta = \eta_0$$

and take the normalization $\sigma : Y \to X$ in $k(X)(\theta, \zeta)$. We then know that Y *is not a uniruled surface*. Furthermore, we have

$$-\eta_0^3 d\theta = dv, \quad -\eta_0^3 d\zeta = d\eta_0$$

on Y. Since $\omega = dv = -\dfrac{s^2}{1+rs} d\eta_0$, we obtain

$$\eta_0^3 d\theta = -\frac{s^2\eta_0^3}{1+rs} d\zeta.$$

By regarding on Y, we have

$$(\omega) = \sigma^*(18\Gamma + 3H).$$

That induces an inclusion

$$\mathscr{O}_Y(\sigma^*(18\Gamma + 3H))\omega \hookrightarrow \mathscr{B}^1_Y.$$

By taking its adjoint, we obtain an injection

$$\mathscr{O}_Y(\sigma^*(6\Gamma + H)) \hookrightarrow F_{Y*}\mathscr{B}^1_{\tilde{Y}}.$$

Moreover, it is a *pre-Tango structure* because $6\Gamma + H$ is ample on X and so is $\sigma^*(6\Gamma + H)$ on Y.

Consider the differential 1-forms $d\theta$ (which is exact) and $t_0 d\theta$ (which is not closed) on Y. We have

$$d\theta = -\frac{s^2}{1+rs} d\zeta \quad \text{and} \quad t_0 d\theta = -\frac{q_0 s}{1+rs} d\zeta.$$

Since $d\theta$, $t_0 d\theta$ are regular on $\sigma^{-1}(V_0)$ and since $\dfrac{s^2}{1+rs} d\zeta$, $\dfrac{q_0 s}{1+rs} d\zeta$ are so on $\sigma^{-1}(W_0)$, we have

$$d\theta,\ t_0 d\theta \in H^0(\sigma^{-1}(\varphi^{-1}(U_0)), \Omega^1_Y) = H^0(U_0, \varphi_*\sigma_*\Omega^1_Y).$$

On the other hand, since $\omega = dv$ is regular on X, we have that dv is regular on $\sigma^{-1}(\varphi^{-1}(U_\infty))$. Hence we obtain that

$$dv \in H^0(\sigma^{-1}(\varphi^{-1}(U_\infty)), \mathscr{B}^1_Y) = H^0(U_\infty, \varphi_* \sigma_* \mathscr{B}^1_Y).$$

Next consider $t_\infty \omega$. We then know that $t_\infty \omega = t_\infty dv$ is regular on V_∞. Besides, since $t_\infty dv = -\dfrac{q_\infty s}{1+rs} d\eta_\infty$, we know that $t_\infty \omega$ is regular on W_∞. By regarding on Y, we have

$$t_\infty dv \in H^0(\sigma^{-1}(\varphi^{-1}(U_\infty)), \Omega^1_Y) = H^0(U_\infty, \varphi_* \sigma_* \Omega^1_Y).$$

Now let us consider $\mathscr{O}_E$-submodules $\mathscr{R}$ and $\mathscr{S}$ of $\varphi_* \sigma_* \Omega^1_Y$ such that

$$\mathscr{R}|_{U_0} = \mathscr{O}_E|_{U_0} d\theta, \qquad \mathscr{R}|_{U_\infty} = \mathscr{O}_E|_{U_\infty} dv,$$

$$\mathscr{S}|_{U_0} = \mathscr{O}_E|_{U_0} d\theta + \mathscr{O}_E|_{U_0} t_0 d\theta, \qquad \mathscr{S}|_{U_\infty} = \mathscr{O}_E|_{U_\infty} dv + \mathscr{O}_E|_{U_\infty} t_\infty dv.$$

Note that the sections of $\mathscr{S}$ which are not contained in $\mathscr{R}$, are non-closed differential 1-forms. Meanwhile, since $-\eta_0^3 d\theta = dv$, we know that $\mathscr{R} \cong \mathscr{O}_E(3P_0)$. Recall that ξ_0, ξ_∞ are corresponding to y, $w/(1+wz)$, respectively. Therefore, the difference $\xi_0 - \xi_\infty$ is corresponding to $1/w^3$. Denote it by $b_{0\infty}$. We then have that $b_{0\infty}$ is a section in $\mathscr{O}_E(U_0 \cap U_\infty)$. Moreover, since $t_i = xy + \xi_i$ for $i = 0, \infty$, we know that $t_0 - t_\infty = b_{0\infty}$. Hence we have

$$-\eta_0^3 t_0 d\theta = t_\infty dv + b_{0\infty} dv$$

and so,

$$(-d\theta, -t_0 d\theta) = (dv, t_\infty dv) \begin{pmatrix} \dfrac{1}{\eta_0^3} & \dfrac{b_{0\infty}}{\eta_0^3} \\ 0 & \dfrac{1}{\eta_0^3} \end{pmatrix}.$$

Therefore, we obtain a short exact sequence

$$0 \to \mathscr{R} \to \mathscr{S} \to \mathscr{O}_E(3P_0) \to 0.$$

Since $H^1(E, \mathscr{R}) \cong H^1(E, \mathscr{O}_E(3P_0)) = 0$ and $H^0(E, \mathscr{O}_E(3P_0)) \neq 0$, we know $H^0(E, \mathscr{R}) \subsetneqq H^0(E, \mathscr{S})$. Hence we conclude that *Y has non-closed global differential 1-forms*.

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