Groups of Russell type and Tango structures

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Abstract. The group of Russell type is a form of the additive group and the Tango structure is a certain invertible sheaf of locally exact differentials on a curve in positive characteristic. By using the notion of Tango structure, we can construct a group of Russell type over a projective curve, whose completion induces some pathological phenomena in positive characteristic. We consider the locus of curves which have such Tango structures, in the moduli space of curves.

1. Groups of Russell type

Let $k$ be a field of positive characteristic which is not perfect. Take an element $a$ in $k$ such that $a \not\in k^p$. Consider a scheme $X_k = \text{Spec} k[x, y]$ subjected to $y^p = x + ax^p$. We know that $X_k$ is a $k$-group scheme with group structures

$$
\begin{align*}
 k[x, y] & \to k[x, y] \otimes_k k[x, y] \\
 x & \mapsto x \otimes 1 + 1 \otimes x \\
 y & \mapsto y \otimes 1 + 1 \otimes y
\end{align*}
$$

and so on.

Take an extension field $k' = k(a^{1/p})$ and consider the base extension $X_{k'} = X_k \otimes_k k'$. We then obtain that $X_{k'}$ is isomorphic to the additive group $\mathcal{G}_{ak'}$ as a $k'$-group scheme. Indeed, we can rewrite

$$y^p = x + ax^p$$

as $(y - a^{1/p}x)^p = x$, so we have $k'[x, y] = k'[t]$, where $t = y - a^{1/p}x$. Namely, we have a diagram

$$
\begin{array}{ccc}
\mathcal{G}_{ak'} & \cong & X_{k'} \\
\downarrow & & \downarrow \\
k' & \to & k,
\end{array}
$$

which is compatible with the group structures.

The following celebrated theorem was given by Russell in 1970:

Key words and phrases. positive characteristic, group of Russell type, Tango structure, pathological phenomena.
Let $L \subset K$ for some field $K \supset k$. We then have that $X$ is isomorphic to a subgroup scheme $\text{Spec} k[x, y]/I$ of $\mathbb{G}_a^2 = \text{Spec} k[x, y]$, where $I = (y^p - a_0 x - a_1 x^p - \ldots - a_m x^{pn})$ with $a_0 \neq 0$ and $m, n \in \mathbb{N}$.

We call such a group scheme a group of Russell type.

2. Tango structures

Let $C$ be a nonsingular projective curve of genus $g > 1$ over an algebraically closed field $k$. For the sake of simplicity, we assume that the characteristic of $k$ is greater than 2, from now on.

Consider the relative Frobenius morphism $F : \tilde{C} \to C$ over $k$ and let $\mathcal{B}$ be the quotient $F_* \mathcal{O}_{\tilde{C}}/\mathcal{O}_C$. In other words, $\mathcal{B}$ is the direct image of the first coboundary of the de Rham complex $F_* \mathcal{B}^\bullet (\Omega^1_{C/k})$. This is a locally free $\mathcal{O}_C$-module of rank $p - 1$.

**Definition 2.1.** We call an invertible subsheaf $\mathcal{L} \subset \mathcal{B}$ a Tango structure if $\mathcal{L}^p \cong \Omega_C$.

Now we suppose that we have a Tango structure $\mathcal{L}$ on $C$. Suppose, furthermore, that there exists an invertible sheaf $\mathcal{N}$ such that $\mathcal{N}^{p-1} \cong \mathcal{L}$. We then have an extension

$$\begin{array}{cccccc}
0 & \to & \mathcal{O}_C & \to & \mathcal{E} & \to & \mathcal{N}^{p-1} & \to & 0 \\
0 & \to & \mathcal{O}_C & \to & F_* \mathcal{O}_{\tilde{C}} & \to & \mathcal{B} & \to & 0.
\end{array}$$

Take an affine open covering $\{U_i\}_{i \in I}$, local generators $1$ and $q_i$ of $\mathcal{E}$ subjected to

$$q_i = b_{ij} + d^{-1}_{ij} q_j,$$

where $\{d_{ij}\}_{i, j \in I}$ are transition matrices of $\mathcal{N}$ and where

$$\left\{ \left( \begin{array}{cc}
1 & b_{ij} \\
0 & d^{-1}_{ij}
\end{array} \right) \right\}_{i, j \in I}$$

are transition functions of $\mathcal{E}$.

Set $a_i = q_i^p$. We know that $a_i = b_{ij}^p + d_{ij}^{p(p-1)} a_j$ and $a_i$ is a local section of $\mathcal{O}_C$.

Consider a group of Russell type

$$X_i = \text{Spec} \mathcal{O}_C(U_i)[x_i, y_i]/(y_i^p - x_i - a_i x_i^p)$$

over each $U_i$.

Let us glue the $X_i$’s together under the relations

$$x_i = d^{-1}_{ij} p x_j, \quad y_i = b_{ij} d^{-1}_{ij} p x_j + d^{-1}_{ij} y_j.$$

Note that we have

$$y_i^p - x_i - a_i x_i^p = (b_{ij} d^{-1}_{ij} p x_j + d^{-1}_{ij} y_j)^p - d^{-1}_{ij} p x_j - (b_{ij}^p + d_{ij}^{p(p-1)} a_j) d^{-1}_{ij} p x_j^p$$

$$= b_{ij}^p d^{-1}_{ij} p x_j^p + d_{ij}^{p(p-1)} a_j d^{-1}_{ij} p x_j^p - d^{-1}_{ij} p x_j^p - b_{ij}^p d^{-1}_{ij} p x_j^p - d_{ij}^{p(p-1)} a_j x_j^p$$

$$= d_{ij}^{p(p-1)} (y_j^p - x_j - a_j x_j^p),$$

which is compatible with the group structures. Thus we obtain a group of Russell type $X$ over $C$.

Indeed, over $\tilde{C}$, we can rewrite

$$y_i^p = x_i + a_i x_i^p$$

as $(y_i - q_i x_i)^p = x_i$. 

Set $t_i = y_i - q_i x_i$. We then obtain $t_i = d^{-1}_{ij} t_j$ and 
\[ X \times_C \tilde{C}_{|U_i} \cong \text{Spec} \mathcal{O}_{\tilde{C}}(U_i)[t_i]. \]
Hence, by considering the vector bundle 
\[ \text{Spec} \text{Symm}(F^* N^{-1}) \to \tilde{C} \]
without scalar-multiplication, we have 
\[ \text{Spec} \text{Symm}(F^* N^{-1}) \cong X \times_C \tilde{C} \to X \]
\[ \downarrow \quad \square \quad \downarrow \]
\[ \tilde{C} \to C. \]

For more generalized discussions on groups of Russell type over a curve, see [13].

3. Completion of $X$

Consider the vector bundle 
\[ \text{Spec} \text{Symm}(N^{-p}) \to C. \]
We have a diagram 
\[ \text{Spec} \text{Symm}(F^* N^{-1}) \to X \to \text{Spec} \text{Symm}(N^{-p}) \]
\[ \downarrow \quad \square \quad \downarrow \quad \checkmark \]
\[ \tilde{C} \to C. \]

The composite of the two top morphisms is the relative Frobenius morphism over $k$. Consider, furthermore, the $\mathbb{P}^1$-bundles 
\[ \text{Proj} \text{Symm}(N^{-p} \oplus \mathcal{O}_C) \to C, \]
\[ \text{Proj} \text{Symm}(F^* N^{-1} \oplus \mathcal{O}_C) \to \tilde{C}, \]
which are completions of $\text{Spec} \text{Symm}(N^{-p}) \to C$ and of $\text{Spec} \text{Symm}(F^* N^{-1}) \to \tilde{C}$, respectively.

Take the normalization $Y$ of $\text{Proj} \text{Symm}(N^{-p} \oplus \mathcal{O}_C)$ in $k(X)$. We then know that $Y$ is a completion of $X$ and obtain a diagram 
\[ \text{Proj} \text{Symm}(F^* N^{-1} \oplus \mathcal{O}_C) \to Y \to \text{Proj} \text{Symm}(N^{-p} \oplus \mathcal{O}_C) \]
\[ \cup \quad \cup \quad \cup \]
\[ \text{Spec} \text{Symm}(F^* N^{-1}) \to X \to \text{Spec} \text{Symm}(N^{-p}) \]
\[ \downarrow \quad \square \quad \downarrow \quad \checkmark \]
\[ \tilde{C} \to C. \]

The composite of the two top morphisms is the relative Frobenius morphism over $k$. 

Here the following theorem, originally given by Raynaud[7], holds:

**Theorem 3.1.** In the same notation and under the same assumption as above, we have that the completion $Y$ is a minimal nonsingular projective surface having a fibration such that each fibre is a rational curve with one cusp of type $u^p + v^{p-1} = 0$. Moreover,

1. If $p = 3$, then $Y$ is a quasi-elliptic surface of $\kappa = 1$.
2. If $p > 3$, then $Y$ is a surface of general type.

In any case, $Y$ gives a counter-example to the Kodaira vanishing theorem in positive characteristic.

For the proof and more detailed discussions on the completion $Y$, see [11, 13].

Needless to say, $Y$ cannot be lifted over the ring of Witt-vectors of length two(see for example [1]). In our case, furthermore, it holds

**Theorem 3.2** (Russell, Ganong, Kurke, W. Lang, Mukai, . . .). Retain the same notation and assumption as above, we have

$$ H^0(Y, \Theta_Y) = H^0(C, N), $$

where $\Theta_Y$ is the tangent sheaf of $Y$.

For the proof, see [2, 3, 4, 6, 9]. See also [12].

These theorems imply that, if $H^0(C, N) \neq 0$ and $p > 3$, then $Y$ is a surface of general type with nontrivial vector fields. On the other hand, it is well-known that the automorphism groups of varieties of general type are finite(see for example [5]). Hence we know that the automorphism group scheme of $Y$ is not reduced.

**Example 3.3.** Consider the plane curve defined by the equation

$$ C : y^{p-1} - x = x^{p(p-1)} $$

in $p > 3$. We then have a Tango structure

$$ L \cong O_C((p-1)(p^2 - p - 3)P_\infty) $$

arising from the exact form $dx$ whose divisor

$$ (dx) = p(p-1)(p^2 - p - 3)P_\infty, $$

where $P_\infty$ is the point at infinity. Let $N = O_C((p^2 - p - 3)P_\infty)$. We then obtain

$$ N^{p-1} \cong L, \quad H^0(C, N) \neq 0. $$

Hence we get a nonsingular projective surface of general type with nontrivial vector fields.

Thus we obtain a really pathological surface in positive characteristic. In the circumstances, the following problems seem interesting:

Are such surfaces really exceptional ones?
Do they form discrete points in the moduli space?

It is, however, likely difficult to solve these problem straightforward, because of the difficulty of the moduli space of surfaces. So, for the first step, let us consider the following problems:

Are the curves having such Tango structures really exceptional ones?
Do they form discrete points in the moduli space?
4. Main theorem

In this last section, we shall give a negative answer to the latter problems. Consider the following locus in the moduli space $M_g$ of curves of genus $g > 1$ such that $p(p-1) \mid 2g - 2$ with $p > 3$:

$$
T_g = \{ \text{C } \in \mathcal{M}_g \mid \text{C has a Tango structure } \mathcal{N}^{p^{-1}} \text{ with } H^0(C, \mathcal{N}) \neq 0 \}.
$$

The main theorem is the following:

**Theorem 4.1** (cf. [14]). The locus $T_g$ contains a variety of dimension $\geq g - 1$ provided $p \equiv 3 \pmod{4}$.

**Proof.** Consider the subvariety

$$
\mathcal{H}_g = \{ \text{C } \in \mathcal{M}_g \mid \text{C is a hyperelliptic curve} \}.
$$

We have a rational mapping

$$
\mathbb{A}^{2g-1} \to \mathcal{H}_g
$$

$$
\bigcup \bigcup
$$

$$(a_1, \ldots, a_{2g-1}) \mapsto y^2 = x(x-1)(x-a_1)\cdots(x-a_{2g-1}).
$$

Let $C$ be an image, that is a hyperelliptic curve, and let

$$
\begin{pmatrix}
 c_{11} & \cdots & c_{1g} \\
 \vdots & \ddots & \vdots \\
 c_{g1} & \cdots & c_{gg}
\end{pmatrix}
$$

be the Hasse-Witt matrix of $C$ with respect to 

$$
\left\{ \frac{dx}{y}, \ldots, \frac{x^{g-1}dx}{y} \right\}.
$$

Note that each entry is a polynomial in $a_1, \ldots, a_{2g-1}$ (see for example [17]). Consider the subvariety $S$ of $\mathbb{A}^{2g-1}$ defined by $c_{11} = \cdots = c_{g1} = 0$, i.e., the first column is zero. We then know that $	ext{dim } S \geq g - 1$ if $S$ is not empty. Let $T$ be the image of $S$ in $\mathcal{H}_g$.

$$
\mathbb{A}^{2g-1} \to \mathcal{H}_g
$$

$$
\bigcup \bigcup
$$

$$
S \longrightarrow T
$$

Since the mapping $\mathbb{A}^{2g-1} \to \mathcal{H}_g$ is induced from an action of a certain finite group, we also know that $\text{dim } T \geq g - 1$ if $T$ is not empty. Moreover, we have that

$$
\mathcal{C}\left(\frac{dx}{y}\right) = 0
$$

on every curve in $T$, where $\mathcal{C}$ is the Cartier operator (see for example [10]). In other words, the form $dx/y$ is exact.

On the other hand, $dx/y$ induces the divisor

$$
\left(\frac{dx}{y}\right) = (2g - 2) P_\infty
$$
and \(2g - 2 = np(p - 1)\) with \(n \in \mathbb{N}\). Therefore, from this exact form, we obtain a Tango structure \(\mathcal{O}_C(nP_\infty)^{p-1} \hookrightarrow \mathcal{B} \) with \(H^0(\mathcal{O}_C(nP_\infty)) \neq 0\).

So we know that \(T\) is contained in the locus \(T_g\).

Now we have only to verify that \(T\) is not empty. Consider the hyperelliptic curve defined by \(y^2 = x^d - x\) with \(d = 2g + 1, 2g - 2 = np(p - 1), n \in \mathbb{N}\). We can then compute as follows:

\[
yC\left(\frac{dx}{y}\right) = C\left(y^{p-1}dx\right) = C((x^d - x)^{\frac{p-1}{2}}dx).
\]

In \((x^d - x)^{\frac{p-1}{2}} = x^{\frac{p-1}{2}}(x^{d-1} - 1)^{\frac{p-1}{2}}\), only terms
\[
x^{\frac{p-1}{2}}, x^{\frac{p-1}{2}+d-1}, x^{\frac{p-1}{2}+2(d-1)}, \ldots, x^{d\frac{p-1}{2}}
\]
appear. More precisely, since \(d = 3 + np(p - 1)\), we know that only terms
\[
x^{\frac{p-1}{2}}, x^{\frac{p-1}{2}+2+np(p-1)}, x^{\frac{p-1}{2}+2+2np(p-1)}, \ldots, x^{\frac{p-1}{2}+\frac{p-1}{2}+np(p-1)}
\]
appear. Note that
\[
\frac{p-1}{2} < p - 1, \quad p < \frac{p-1}{2} + \frac{p-1}{2} < 2p - 1.
\]

We can verify that \(C(dx/y) \neq 0\) if and only if there exists an integer \(l\) such that
\[
\frac{p-1}{2} + 2l = p - 1 \quad \text{with} \quad 1 \leq l \leq \frac{p-1}{2},
\]
i.e., \(4l = p - 1\). In other words,
\[
C\left(\frac{dx}{y}\right) \neq 0 \quad \text{if and only if} \quad 4 \mid p - 1.
\]

Otherwise \(dx/y\) is exact and so we know that our curve is lying in \(T\). That is the required assertion. \(\Box\)

**Remark 4.2.** By applying Tsuda’s method([16]), we get a slightly better estimate. Retain the same notation and assumption as in the previous theorem. Consider a hyperelliptic curve defined by

\[
y^2 = x(x-1)(x-a_1) \cdots (x-a_{2g-1})
\]
and the differential form \(dx/y\). We can compute as follows:

\[
yC\left(\frac{dx}{y}\right) = C(y^{p-1}dx) = C((x(x-1)(x-a_1) \cdots (x-a_{2g-1}))^{\frac{p-1}{2}}dx).
\]

Let

\[
(x(x-1)(x-a_1) \cdots (x-a_{2g-1}))^{\frac{p-1}{2}} = b_{\frac{p-1}{2}}x^{\frac{p-1}{2}} + b_{\frac{p-1}{2}+1}x^{\frac{p-1}{2}+1} + \cdots + b_{\frac{p-1}{2}-1}x^{\frac{p-1}{2}-1} + x^d.
\]

Note that each \(b_i\) is a polynomial in \(a_1, \ldots, a_{2g-1}\). Since \(d = 2g + 1 = np(p - 1) + 3\), we know

\[
d\frac{p-1}{2} = \frac{n(p-1)^2}{2}p + 3\frac{p-1}{2}.
\]
It follows that we have $C(dx/y) = 0$ if the coefficients

$$b_{p-1}, b_{2p-1}, \ldots, b_{\left(\frac{n(p-1)^2}{2}+1\right)p-1}$$

are zero. Consider the subvariety of $\mathbb{A}^{2g-1}$ defined by

$$b_{p-1} = b_{2p-1} = \cdots = b_{\left(\frac{n(p-1)^2}{2}+1\right)p-1} = 0$$

and let $U$ be its image in $\mathcal{T}_g$. We then know that $U$ is contained in $\mathcal{T}_g$. Moreover, the number of the above-mentioned coefficients is $\frac{n(p-1)^2}{2} + 1$, which is less than $g = \frac{np(p-1)}{2} + 1$.

On the other hand, the hyperelliptic curve defined by $y^2 = x^d - x$, mentioned in the previous proof, is lying in $U$ provided $4 \nmid p - 1$. Hence we obtain

$$\dim U \geq 2g - \frac{n(p-1)^2}{2} > g - 1.$$ 

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References


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