Groups of Russell type and Tango structures

Yoshifumi TAKEDA

Abstract. The group of Russell type is a form of the additive group and the Tango structure is a certain invertible sheaf of locally exact differentials on a curve in positive characteristic. By using the notion of Tango structure, we can construct a group of Russell type over a projective curve, whose completion induces some pathological phenomena in positive characteristic. We consider the locus of curves which have such Tango structures, in the moduli space of curves.

1. Groups of Russell type

Let k be a field of positive characteristic which is not perfect. Take an element a in k such that $a \notin k^p$. Consider a scheme $X_k = \operatorname{Spec} k[x, y]$ subjected to $y^p = x + ax^p$. We know that X_k is a k-group scheme with group structures

$$\begin{array}{ccc} k[x,y] & \to & k[x,y] \otimes_k k[x,y] \\ x & \mapsto & x \otimes 1 + 1 \otimes x \\ y & \mapsto & y \otimes 1 + 1 \otimes y \end{array}$$

and so on.

Take an extension field $k' = k(a^{1/p})$ and consider the base extension

$$X_{k'} = X_k \otimes_k k'.$$

We then obtain that $X_{k'}$ is isomorphic to the additive group $\mathbb{G}_{ak'}$ as a k'-group scheme. Indeed, we can rewrite

$$y^p = x + ax^p \quad \text{as} \quad (y - a^{1/p}x)^p = x,$$

so we have $k'[x, y] = k'[t]$, where $t = y - a^{1/p}x$. Namely, we have a diagram

$$\begin{array}{ccccc} \mathbb{G}_{ak'} & \cong & X_{k'} & \to & X_k \\ & & \downarrow & \square & \downarrow \\ & & k' & \to & k, \end{array}$$

which is compatible with the group structures.

The following celebrated theorem was given by Russell in 1970:

Key words and phrases. positive characteristic, group of Russell type, Tango structure, pathological phenomena.

THEOREM 1.1 ([**8**]). *Let X be a k-group scheme such that $X \otimes_k K \cong \mathbb{G}_{aK}$ for some field $K \supset k$. We then have that X is isomorphic to a subgroup scheme* $\operatorname{Spec} k[x,y]/I$ *of* $\mathbb{G}_{ak}^2 = \operatorname{Spec} k[x,y]$, *where* $I = (y^{p^n} - a_0x - a_1x^p - \cdots - a_mx^{p^m})$ *with* $a_0 \neq 0$ *and* $m, n \in \mathbb{N}$.

We call such a group scheme a *group of Russell type*.

2. Tango structures

Let C be a nonsingular projective curve of genus $g > 1$ over an algebraically closed field k. For the sake of simplicity, we assume that the characteristic of k is greater than 2, from now on.

Consider the relative Frobenius morphism $F : \widetilde{C} \to C$ over k and let $\mathcal{B}$ be the quotient $F_*\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C$. In other words, $\mathcal{B}$ is the direct image of the first coboundary of the de Rham complex $F_*\mathrm{B}^1(\Omega^\bullet_{\widetilde{C}})$. This is a locally free $\mathcal{O}_C$-module of rank $p-1$.

DEFINITION 2.1. We call an invertible subsheaf $\mathcal{L} \subset \mathcal{B}$ a *Tango structure* if $\mathcal{L}^p \cong \Omega_C$.

Now we suppose that we have a Tango structure $\mathcal{L}$ on C. Suppose, furthermore, that there exists an invertible sheaf $\mathcal{N}$ such that $\mathcal{N}^{p-1} \cong \mathcal{L}$. We then have an extension

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_C & \to & \mathcal{E} & \to & \mathcal{N}^{p-1} & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & \mathcal{O}_C & \to & F_*\mathcal{O}_{\widetilde{C}} & \to & \mathcal{B} & \to & 0. \end{array}$$

Take an affine open covering $\{U_i\}_{i \in I}$, local generators 1 and q_i of $\mathcal{E}$ subjected to

$$q_i = b_{ij}\,1 + d_{ij}^{p-1}q_j,$$

where $\{d_{ij}\}_{i,j \in I}$ are transition functions of $\mathcal{N}$ and where

$$\left\{ \begin{pmatrix} 1 & b_{ij} \\ 0 & d_{ij}^{p-1} \end{pmatrix} \right\}_{i,j \in I}$$

are transition matrices of $\mathcal{E}$.

Set $a_i = q_i^p$. We know that $a_i = b_{ij}^p + d_{ij}^{p(p-1)}a_j$ and a_i is a local section of $\mathcal{O}_C$. Consider a group of Russell type

$$X_i = \operatorname{Spec} \mathcal{O}_C(U_i)[x_i, y_i]/(y_i^p - x_i - a_ix_i^p)$$

over each U_i.

Let us glue the X_i's together under the relations

$$x_i = d_{ij}^{-p}x_j, \qquad y_i = b_{ij}d_{ij}^{-p}x_j + d_{ij}^{-1}y_j.$$

Note that we have

$$\begin{aligned} y_i^p - x_i - a_ix_i^p &= (b_{ij}d_{ij}^{-p}x_j + d_{ij}^{-1}y_j)^p - d_{ij}^{-p}x_j - (b_{ij}^p + d_{ij}^{p(p-1)}a_j)d_{ij}^{-p^2}x_j^p \\ &= b_{ij}^pd_{ij}^{-p^2}x_j^p + d_{ij}^{-p}y_j^p - d_{ij}^{-p}x_j - b_{ij}^pd_{ij}^{-p^2}x_j^p - d_{ij}^{-p}a_jx_j^p \\ &= d_{ij}^{-p}(y_j^p - x_j - a_jx_j^p), \end{aligned}$$

which is compatible with the group structures. Thus we obtain a *group of Russell type X over C*.

Indeed, over $\widetilde{C}$, we can rewrite

$$y_i^p = x_i + a_ix_i^p \quad \text{as} \quad (y_i - q_ix_i)^p = x_i.$$

Set $t_i = y_i - q_i x_i$. We then obtain $t_i = d_{ij}^{-1} t_j$ and

$$X \times_C \widetilde{C}|_{U_i} \cong \operatorname{Spec} \mathcal{O}_{\widetilde{C}}(U_i)[t_i].$$

Hence, by considering the vector bundle

$$\mathbb{S}\mathrm{pec}\, \mathit{Symm}(F^*\mathcal{N}^{-1}) \to \widetilde{C}$$

without scalar-multiplication, we have

$$\begin{tikzcd}
\mathbb{S}\mathrm{pec}\, \mathit{Symm}(F^*\mathcal{N}^{-1}) \arrow[r, phantom, "\cong"] \arrow[dr] & X \times_C \widetilde{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
& \widetilde{C} \arrow[r, "F"'] & C.
\end{tikzcd}$$

For more generalized discussions on groups of Russell type over a curve, see [**13**].

3. Completion of X

Consider the vector bundle

$$\mathbb{S}\mathrm{pec}\, \mathit{Symm}(\mathcal{N}^{-p}) \to C.$$

We have a diagram

$$\begin{tikzcd}
\mathbb{S}\mathrm{pec}\, \mathit{Symm}(F^*\mathcal{N}^{-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[r] \arrow[d] & \mathbb{S}\mathrm{pec}\, \mathit{Symm}(\mathcal{N}^{-p}) \arrow[dl] \\
\widetilde{C} \arrow[r] & C. &
\end{tikzcd}$$

The composite of the two top morphisms is the relative Frobenius morphism over k. Consider, furthermore, the $\mathbb{P}^1$-bundles

$$\mathbb{P}\mathrm{roj}\, \mathit{Symm}(\mathcal{N}^{-p} \oplus \mathcal{O}_C) \to C,$$
$$\mathbb{P}\mathrm{roj}\, \mathit{Symm}(F^*\mathcal{N}^{-1} \oplus \mathcal{O}_{\widetilde{C}}) \to \widetilde{C},$$

which are completions of $\mathbb{S}\mathrm{pec}\, \mathit{Symm}(\mathcal{N}^{-p}) \to C$ and of $\mathbb{S}\mathrm{pec}\, \mathit{Symm}(F^*\mathcal{N}^{-1}) \to \widetilde{C}$, respectively.

Take the normalization Y of $\mathbb{P}\mathrm{roj}\, \mathit{Symm}(\mathcal{N}^{-p} \oplus \mathcal{O}_C)$ in $k(X)$. We then know that Y is a completion of X and obtain a diagram

$$\begin{tikzcd}
\mathbb{P}\mathrm{roj}\, \mathit{Symm}(F^*\mathcal{N}^{-1} \oplus \mathcal{O}_{\widetilde{C}}) \arrow[r] & Y \arrow[r] & \mathbb{P}\mathrm{roj}\, \mathit{Symm}(\mathcal{N}^{-p} \oplus \mathcal{O}_C) \\
\cup & \cup & \cup \\
\mathbb{S}\mathrm{pec}\, \mathit{Symm}(F^*\mathcal{N}^{-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[r] \arrow[d] & \mathbb{S}\mathrm{pec}\, \mathit{Symm}(\mathcal{N}^{-p}) \arrow[dl] \\
\widetilde{C} \arrow[r] & C. &
\end{tikzcd}$$

The composite of the two top morphisms is the relative Frobenius morphism over k.

Here the following theorem, originally given by Raynaud[**7**], holds:

THEOREM 3.1. *In the same notation and under the same assumption as above, we have that the completion Y is a minimal nonsingular projective surface having a fibration such that each fibre is a rational curve with one cusp of type $u^p + v^{p-1} = 0$. Moreover,*

(1) *If $p = 3$, then Y is a quasi-elliptic surface of $\kappa = 1$.*
(2) *If $p > 3$, then Y is a surface of general type.*

In any case, Y gives a counter-example to the Kodaira vanishing theorem in positive characteristic.

For the proof and more detailed discussions on the completion Y, see [**11, 13**].

Needless to say, Y cannot be lifted over the ring of Witt-vectors of length two(see for example [**1**]). In our case, furthermore, it holds

THEOREM 3.2 (Russell, Ganong, Kurke, W. Lang, Mukai, ...). *Retain the same notation and assumption as above, we have*

$$H^0(Y, \Theta_Y) = H^0(C, \mathcal{N}),$$

where Θ_Y is the tangent sheaf of Y.

For the proof, see [**2, 3, 4, 6, 9**]. See also [**12**].

These theorems imply that, if $H^0(C, \mathcal{N}) \neq 0$ and $p > 3$, then Y is a surface *of general type with nontrivial vector fields.* On the other hand, it is well-known that the automorphism groups of varieties of general type are finite(see for example [**5**]). Hence we know that the automorphism group scheme of Y is not reduced.

EXAMPLE 3.3. Consider the plane curve defined by the equation

$$C : y^{p(p-1)} - y = x^{p(p-1)-1}$$

in $p > 3$. We then have a Tango structure

$$\mathcal{L} \cong \mathcal{O}_C((p-1)(p^2 - p - 3)P_\infty)$$

arising from the exact form dx whose divisor

$$(dx) = p(p-1)(p^2 - p - 3)P_\infty,$$

where P_∞ is the point at infinity. Let $\mathcal{N} = \mathcal{O}_C((p^2 - p - 3)P_\infty)$. We then obtain

$$\mathcal{N}^{p-1} \cong \mathcal{L}, \quad H^0(C, \mathcal{N}) \neq 0.$$

Hence we get a nonsingular projective surface of general type with nontrivial vector fields.

Thus we obtain a really pathological surface in positive characteristic. In the circumstances, the following problems seem interesting:

Are such surfaces really exceptional ones?
Do they form discrete points in the moduli space?

It is, however, likely difficult to solve these problem straightforward, because of the difficulty of the moduli space of surfaces. So, for the first step, let us consider the following problems:

Are the curves having such Tango structures really exceptional ones?
Do they form discrete points in the moduli space?

4. Main theorem

In this last section, we shall give a negative answer to the latter problems. Consider the following locus in the moduli space $\mathcal{M}_g$ of curves of genus $g > 1$ such that $p(p-1) \mid 2g-2$ with $p > 3$:

$$\mathcal{T}_g = \left\{ C \in \mathcal{M}_g \;\middle|\; C \text{ has a Tango structure } \mathcal{N}^{p-1} \text{ with } H^0(C, \mathcal{N}) \neq 0 \right\}.$$

The main theorem is the following:

THEOREM 4.1 (cf. [**14**]). *The locus* $\mathcal{T}_g$ *contains a variety of* $\dim \geq g-1$ *provided* $p \equiv 3 \pmod 4$.

PROOF. Consider the subvariety

$$\mathcal{H}_g = \{ C \in \mathcal{M}_g \mid C \text{ is a hyperelliptic curve} \}.$$

We have a rational mapping

$$\begin{array}{ccc} \mathbb{A}^{2g-1} & \to & \mathcal{H}_g \\ \Cup & & \Cup \\ (a_1, \ldots, a_{2g-1}) & \mapsto & y^2 = x(x-1)(x-a_1)\cdots(x-a_{2g-1}). \end{array}$$

Let C be an image, that is a hyperelliptic curve, and let

$$\begin{pmatrix} c_{11} & \cdots & c_{1g} \\ \vdots & & \vdots \\ c_{g1} & \cdots & c_{gg} \end{pmatrix}$$

be the Hasse-Witt matrix of C with respect to

$$\left\{ \frac{dx}{y}, \ldots, \frac{x^{g-1}dx}{y} \right\}.$$

Note that each entry is a polynomial in $a_1, \cdots, a_{2g-1}$ (see for example [**17**]). Consider the subvariety S of $\mathbb{A}^{2g-1}$ defined by $c_{11} = \cdots = c_{g1} = 0$, i.e., the first column is zero. We then know that $\dim S \geq g-1$ if S is not empty. Let T be the image of S in $\mathcal{H}_g$.

$$\begin{array}{ccc} \mathbb{A}^{2g-1} & \to & \mathcal{H}_g \\ \cup & & \cup \\ S & \longrightarrow & T \end{array}$$

Since the mapping $\mathbb{A}^{2g-1} \to \mathcal{H}_g$ is induced from an action of a certain finite group, we also know that $\dim T \geq g-1$ if T is not empty. Moreover, we have that

$$\mathcal{C}\left(\frac{dx}{y}\right) = 0$$

on every curve in T, where $\mathcal{C}$ is the Cartier operator(see for example [**10**]). In other words, the form dx/y is exact.

On the other hand, dx/y induces the divisor

$$\left(\frac{dx}{y}\right) = (2g-2)P_\infty$$

and $2g-2=np(p-1)$ with $n\in\mathbb{N}$. Therefore, from this exact form, we obtain a Tango structure

$$\mathcal{O}_C(nP_\infty)^{p-1}\hookrightarrow\mathcal{B}\quad\text{with}\quad H^0(\mathcal{O}_C(nP_\infty))\neq 0.$$

So we know that T is contained in the locus $\mathcal{T}_g$.

Now we have only to verify that T is not empty. Consider the hyperelliptic curve defined by

$$y^2=x^d-x$$

with $d=2g+1$, $2g-2=np(p-1)$, $n\in\mathbb{N}$. We can then compute as follows:

$$y\,\mathcal{C}\Big(\frac{dx}{y}\Big)=\mathcal{C}(y^{p-1}dx)=\mathcal{C}((x^d-x)^{\frac{p-1}{2}}dx).$$

In $(x^d-x)^{\frac{p-1}{2}}=x^{\frac{p-1}{2}}(x^{d-1}-1)^{\frac{p-1}{2}}$, only terms

$$x^{\frac{p-1}{2}},\ x^{\frac{p-1}{2}+d-1},\ x^{\frac{p-1}{2}+2(d-1)},\dots,\ x^{d\frac{p-1}{2}}$$

appear. More precisely, since $d=3+np(p-1)$, we know that only terms

$$x^{\frac{p-1}{2}},\ x^{\frac{p-1}{2}+2+np(p-1)},\ x^{\frac{p-1}{2}+2\cdot 2+2np(p-1)},\ \dots,\ x^{\frac{p-1}{2}+\frac{p-1}{2}2+\frac{p-1}{2}np(p-1)}$$

appear. Note that

$$\frac{p-1}{2}<p-1,\quad p<\frac{p-1}{2}+\frac{p-1}{2}2<2p-1.$$

We can verify that $\mathcal{C}(dx/y)\neq 0$ if and only if there exists an integer l such that

$$\frac{p-1}{2}+2l=p-1\quad\text{with}\quad 1\le l\le\frac{p-1}{2},$$

i.e., $4l=p-1$. In other words,

$$\mathcal{C}\Big(\frac{dx}{y}\Big)\neq 0\quad\text{if and only if}\quad 4\mid p-1.$$

Otherwise dx/y is exact and so we know that our curve is lying in T. That is the required assertion. □

Remark 4.2. By applying Tsuda's method([**16**]), we get a slightly better estimate. Retain the same notation and assumption as in the previous theorem. Consider a hyperelliptic curve defined by

$$y^2=x(x-1)(x-a_1)\cdots(x-a_{2g-1})$$

and the differential form dx/y. We can compute as follows:

$$y\,\mathcal{C}\Big(\frac{dx}{y}\Big)=\mathcal{C}(y^{p-1}dx)=\mathcal{C}((x(x-1)(x-a_1)\cdots(x-a_{2g-1}))^{\frac{p-1}{2}}dx).$$

Let

$$\begin{aligned}&(x(x-1)(x-a_1)\cdots(x-a_{2g-1}))^{\frac{p-1}{2}}\\ &\qquad=b_{\frac{p-1}{2}}\,x^{\frac{p-1}{2}}+b_{\frac{p-1}{2}+1}\,x^{\frac{p-1}{2}+1}+\cdots+b_{d\frac{p-1}{2}-1}\,x^{d\frac{p-1}{2}-1}+x^{d\frac{p-1}{2}}.\end{aligned}$$

Note that each b_i is a polynomial in $a_1,\ \dots,\ a_{2g-1}$. Since $d=2g+1=np(p-1)+3$, we know

$$d\frac{p-1}{2}=\frac{n(p-1)^2}{2}p+3\frac{p-1}{2}.$$

It follows that we have $\mathcal{C}(dx/y) = 0$ if the coefficients

$$b_{p-1},\ b_{2p-1},\ \ldots,\ b_{(\frac{n(p-1)^2}{2}+1)p-1}$$

are zero. Consider the subvariety of $\mathbb{A}^{2g-1}$ defined by

$$b_{p-1} = b_{2p-1} = \cdots = b_{(\frac{n(p-1)^2}{2}+1)p-1} = 0$$

and let U be its image in $\mathcal{H}_g$. We then know that U is contained in $\mathcal{T}_g$. Moreover, the number of the above-mentioned coefficients is $\dfrac{n(p-1)^2}{2} + 1$, which is less than $g = \dfrac{np(p-1)}{2} + 1$.

On the other hand, the hyperelliptic curve defined by $y^2 = x^d - x$, mentioned in the previous proof, is lying in U provided $4 \not| p-1$. Hence we obtain

$$\dim U \geq 2g - \frac{n(p-1)^2}{2} > g - 1.$$

Acknowledgements

The author expresses his sincere gratitude to the organizers of the conference for their invitation.

References

[1] P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), 247–270.

[2] R. Ganong and P. Russell, *The tangent bundle of a ruled surface*, Math. Ann. **271** (1985), 527–548.

[3] H. Kurke, *Example of false ruled surfaces*, in Proceedings of Symposium on Algebraic Geometry Kinosaki, 1981, pp. 203–223.

[4] W. E. Lang, *Examples of surfaces of general type with vector fields*, in Arithmetic and Geometry, Vol. II, M. Artin, J. Tate (eds.), Birkhäuser, Boston, 1983, pp. 167–173.

[5] H. Matsumura, *On algebraic groups of birational transformations*, Atti Accad. Naz. dei Lincei, **34**, (1963), 151-155.

[6] S. Mukai, *On counter-examples to the Kodaira vanishing theorem and the Yau inequality in positive characteristic* (Japanese), in Proceedings of Symposium on Algebraic Geometry Kinosaki, 1979, pp. 9–31.

[7] M. Raynaud, *Contre-exemple au "Vanishing Theorem" em caractéristique $p > 0$*, in C. P. Ramanujan-A Tribute, Tata Inst. Fund. Res. Studies in Math. 8, Springer-Verlag, Berlin, Heidelberg, New York, 1978, pp.273–278.

[8] P. Russell, *Forms of the affine line and its additive group*, Pacific J. Math. **32**, (1970), 527–539.

[9] P. Russell, *Factoring the Frobenius morphism of an algebraic surface*, in Algebraic Geometry, Bucharest 1982 (Bucharest, 1982), Lecture Notes in Math. 1056, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984, pp. 366–380.

[10] J.-P. Serre, *Sur la topologie des variétés algébriques en caracteristique p*, in Symposium Internacional de Topologia Algebraica, Mexico, 1958, pp. 24–53.

[11] Y. Takeda, *Fibrations with moving cuspidal singularities*, Nagoya Math. J. **122**, (1991), 161–179.

[12] Y. Takeda, *Vector fields and differential forms on generalized Raynaud surfaces*, Tôhoku Math. J. **44**, (1992), 359–364.

[13] Y. Takeda, *Groups of Russell type over a curve*, J. Pure and Applied Algebra **128**, (1998), 93–108; Corrigendum, **148**, (2000), 317–318.

[14] Y. Takeda and K. Yokogawa, *Pre-Tango structures on curves*, Tôhoku Math. J. **54**, (2002), 227–237; Errata and addenda, **55**, (2003), 611–614.

[15] H. Tango, *On the behavior of extensions of vector bundles under the Frobenius map*, Nagoya Math. J. **48**, (1972), 73–89.
[16] N. Tsuda, *Pre-Tango structures on hyperelliptic cures* (Japanese), Master's Thesis, Osaka University, Osaka, 2003.
[17] N. Yui, *On the Jacobian varieties of hyperelliptic curves over fields of characteristic* $p > 2$, J. Algebra, **52**, (1978), 378–410.

Department of Mathematics and Statistics, Wakayama Medical University, Wakayama City 6418509, JAPAN